Performance of Auctions and Sealed Bids

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Abstract. We develop models of automated E-commerce techniques, which predict the economic outcomes of these decision mechanisms, including the price attained by a good and the resulting income per unit time, as a function of the rate at which bidders provide the bids and of the time taken by the seller to decide whether to accept a bid. This paper extends previous work in two main directions. Since automated E-commerce mechanisms are typically implemented in software residing on the Internet, this paper shows how network quality of service will impact the economic outcome of automated auctions. We also analyse sealed bids which can also be automated, but which differ significantly from auctions in the manner in which information is shared between the bidders and the the party that decides the outcome. The approach that we propose opens novel avenues of research that bring together traditional computer and system performance analysis and the economic analysis of Internet based trading methods.

Keywords: Auctions, Sealed Bids, Networked Economics, Economic Performance, Quality of Service.

1 Introduction

Recent events in finance and the stock market have shown that our understanding of the automated computer and network based economy is not sufficiently advanced to be able to predict the dramatic changes that occur in the price of securities, commodities and the overall behaviour of prices. Thus more analysis is needed about how prices are established in the presence of automated

⋆ Research undertaken as part of the ALADDIN (Autonomous Learning Agents for Decentralised Data and Information Systems) project, jointly funded by a BAE Systems and EPSRC (Engineering and Physical Research Council) strategic partnership under Grant No. EP/C548051/1.

⋆⋆ Supported by the Computer and Automation Research Institute of the Hungarian Academy of Sciences.

J.T. Bradley (Ed.): EPEW 2009, LNCS 5652, pp. 30–43, 2009.

trading patterns. We feel that it would be presumptuous to try to model such systems in general and we propose to begin from modest premises and simple models, trying to build up to large systemic representations from solid building blocks based on first principles. This paper pursues the study of such systems using methods which has been successfully used in the performance analysis of computer systems and networks, and in operations research.

Among the techniques used in commerce and E-commerce, auctions have been the subject of some investigation [3]. They have also been used as tools for decision making in resource allocation problems [13] and for the coordination of software agents. The advantage of auctions is that the mechanisms that they use for negotiation between economic agents, and for decision, are clearly defined so that precise models of the resulting behaviour can be constructed. Models of auctions and sealed bids are amenable to rigorous analysis, as shown in early work on Martin Gardner's "Secretary, or Sultan's Dowry, Problem" where a recruiter selects the best candidate from a sequence of applicants, the quality of successive candidates are random variables, and the recruiter must make the irrevocable choice of a candidate from an initial sequence, without the possibility of further candidates being considered [1, 11] after the decision is made. Other analysis of auctions can be found in [8–10, 14, 15]. Of course there also direct links between auctions and networks due to the sale of wireless spectrum; however virtual auctions have also been suggested as a means of allocating network bandwidth [18] and the wireless spectrum, in real time, to competing users [19].

In this paper we pursue the work begun in [17, 20] by studying price formation in auctions. In simple terms, the analysis aims at obtaining the economic performance of an auction in terms of the price that is attained, which is of interest to the buyers, and the income per unit time, which is of interest to sellers. This paper first recalls some of the earlier work in [17], and then shows how it can be extended in several directions, and in particular to:

- (i) Include the effect of network QoS on the economic performance criteria, and in particular the impact of packet or message loss and delay.
- (ii) Discuss the effect of items which may not be available for sale, or whose arrival to the auction may be delayed.
- (iii) Economic mechanisms which differ from auctions, such as sealed bidding schemes, where (contrary to auctions) the value of the offers are not known to everyone until *all* the offers have been made, and a given deadline or time-out has expired.

In the following sections we first discuss the probabilistic structure that we use has inter-arrival times for bids, random variables for the increments or values of successive bids, and a probabilistic representation of the time that the seller takes to make his decision after a bid is received. We then compute measures of interest such as the income per unit time resulting from repeated auctions, and the price attained by the good being sold. Since we focus on systems that are built in software on top of computer networks, we analyse the impact that the network quality of service (QoS), such as packet losses and delays, will have on the economic outcome of the auction. Finally, we study similar models for sealed bids, where the successive bidders may not be aware of the price attained by the preceding bids. Thus our work also provides a handle to examining the links between economic performance and the performance of the underlying computer networks that support this economic activity, and we hope that this paper can provide a novel link between trading systems that reside on the Internet and computer system and network performance evaluation, combining traditional performance analysis together with the economic performance at the application level.

2 Analysis of Auctions with Competitive Bids

As mentioned in the introduction, auctions are very common and many will have a direct personal experience of such mechanisms. In this section we focus on English auctions with a known fixed *reserve price* s so that the bidding process only really begins when a bid arrives that exceeds this price. The notion of reserve price is that the seller will only sell the good if the bid is higher than s which is known in advance both to the bidders and to the seller.

As with all English auctions, successive bidders have to make a bid which is *higher* than the previous bid, and the bidder cannot renege: i.e. after a bid is made, if the seller accepts it then the price is that of the highest bid. We also consider an auction centre which is running many successive auctions, which we model as unlimited sequence of statistically identical auctions, with a rest period between successive auctions. This approach will also allow us to compute the relevant quantities for a single auction, i.e. the first and last one, if auctions do not repeat themselves. We assume that the first auction starts at time $t = 0$, and that bids arrive at random times $0 < T_1 < T_2 < \ldots$. Denote by B_i the bid generated at T_i. The $(i+1)th$ bid is of value $B_{i+1} = B_i + I_{i+1}$ where the increment I_{i+1} is a positive random variable representing the increment over the previous bid, while the first bid needs to be equal to the reserve price $B_1 = s$.

Decisions on the part of the seller during the first auction are represented by the *decision delays* $D_i \geq 0$ which are random variables. If $T_{i+1} < T_i + D_i$ then the ith bid is superseded by the $(i+1)th$ bid, while when $T_{i+1} \geq T_i + D_i$, then the seller accepts the ith bid at instant $T_i + D_i$ and the first auction ends at that instant. Thus the $1st$ auction ends at the Nth bid where N is the positive random variable:

$$N = \min\{i : T_i + D_i \leq T_{i+1}\} \tag{1}$$

and hence the *seller's income* brought by this sale is:

$$S = B_N \tag{2}$$

After this first sale is completed, the administrative details required by the sale will take another time duration $R \geq 0$, so the duration of the auction is

$$\tau = R + T_N + D_N. \tag{3}$$

We have to calculate the expected income $\mathbf{E}\{B_N\}$, and the expected duration of the auction $\mathbf{E}\{R + T_N + D_N\}$. Once the first auction period is over, the whole auction repeats itself indefinitely and independently of the previous auction. Let (S_i, τ_i) denote the pair of the income and duration of the *ith* auction, so that after the *nth* auction we have the total *income per unit time* resulting from this sequence of auctions is:

$$\phi_n = \frac{\sum_{i=1}^{n} S_i}{\sum_{i=1}^{n} \tau_i}.$$

By the strong law of large numbers, we have that

$$\phi_n = \frac{\frac{1}{n}\sum_{i=1}^{n} S_i}{\frac{1}{n}\sum_{i=1}^{n} \tau_i} \to \phi = \frac{\mathbf{E}\{S\}}{\mathbf{E}\{\tau\}} = \frac{\mathbf{E}\{B_N\}}{\mathbf{E}\{R + T_N + D_N\}}. \tag{4}$$

In order to characterize the process that we have just described, some assumptions will be needed about these random variables. We characterize the assumptions as follows: assume that $\{(T_i, B_i, D_i)\}_{i=1}^{\infty}$ is a Markov process such that the transition probabilities have a special form:

$$\begin{aligned}
&\mathbf{P}\{T_{i+1} - T_i \le t, I_{i+1} \le u, D_{i+1} \le d \mid B_i = b\} \\
&= \mathbf{P}\{T_{i+1} - T_i \le t \mid B_i = b\} \cdot \mathbf{P}\{I_{i+1} \le u \mid B_i = b\} \cdot \mathbf{P}\{D_{i+1} \le d \mid B_i = b\} \\
&= A(t \mid b) \cdot I(u \mid b) \cdot D(d \mid b),
\end{aligned}$$

so that we recognise the fact that the increments I_{i+1} will depend on the price that has been attained in the most recent bid $B_i = b$, the inter-arrival time of the bids $T_{i+1} - T_i$ and the decision delay D_{i+1} also depend on the value of the most recent bid $B_i = b$; indeed as the price of the bid increases, we may expect that new bids arrive more slowly while the seller may also become more anxious to sell. Finally we assume that R is a random variable with some general distribution function

$$R(t) = \mathbf{P}\{R \le t\}$$

and that it is independent of all the preceding random variables. It is the model of the situation, where there is an infinite buyers' population, and each buyer makes bids sequentially such that they have feedback information from the seller on the bids, moreover each buyer knows the initial and end time points of the auction period.

2.1 The Case with Markovian Bid Arrivals, Exponential Decision Times, and Unit Increments

A special case of the model we have described was analyzed in [17] under the assumption of unit increments for successive bids $I_{i+1} = B_{i+1} - B_i = +1$ $(i \ge 1)$, while $B_1 = s$. That analysis also assumed that the bids' successive interarrival times $T_{i+1} - T_i$ are independent and exponentially distributed with parameter λ_b $(i \ge 1)$, dependent on b the value of the most recent bid, and that T_1 has exponential distribution with parameter λ_0. The decision times D_{i+1} are also mutually independent and exponentially distributed with parameter δ_b dependent on the value b of the most recent bid. Assume also that the rest periods R

are exponentially distributed with average value 1, so that all times and rates have been normalized to the unit value of the rest period.

Under these assumption the price S_t is a birth process, therefore we construct a state-space, where state 0 is the state in which the auction has started or restarted and the seller is waiting for the $1st$ bid, state l is the state in which the current bid is at level $l \geq s$, where s is as before, the "reserve price" or the minimum acceptable level for a bid. Finally $a(l)$ is state in which the bid of level l has been accepted, and the auction 'rests" for a time whose average value is 1 before entering the 0 state where the auction starts again. In the steady-state the corresponding probabilities $p(0),\ p(l),\ p(a(l))$ will satisfy the following balance equations:

$$[\lambda_l + \delta_l]p(l) = \lambda_{l-1}p(l-1),\, l > s > 0,$$

and

$$[\lambda_s + \delta_s]p(s) = \lambda_0 p(0),$$

and

$$p(a(l)) = \delta_l p(l), l \geq s > 0,$$

and

$$\lambda_0 p(0) = \sum_{l=s}^{\infty} p(a(l)),$$

so that

$$p(0) = \sum_{l=s}^{\infty} \frac{\delta_l}{\lambda_0} p(l).$$

These equations together with the condition that the sum of the probabilities is one, give us:

$$p(l) = p(s) \prod_{k=s+1}^{l} \frac{\lambda_{k-1}}{\lambda_k + \delta_k}, l > s,$$

$$1 = p(0) + \sum_{l=s}^{\infty} [p(l) + p(a(l))] = p(0) + \sum_{l=s}^{\infty} p(l)[1 + \delta_l],$$

$$1 = \frac{\delta_s}{\lambda_0} p(s) + p(s)(1 + \delta_s) + p(s) \sum_{l=s+1}^{\infty} \left[1 + \delta_l + \frac{\delta_l}{\lambda_0}\right] \prod_{k=s+1}^{l} \frac{\lambda_{k-1}}{\lambda_k + \delta_k}$$

so that:

$$p(s) = \left[1 + \frac{\delta_s}{\lambda_0} + \delta_s + \sum_{l=s+1}^{\infty} [1 + \delta_l + \frac{\delta_l}{\lambda_0}] \prod_{k=s+1}^{l} \frac{\lambda_{k-1}}{\lambda_k + \delta_k}\right]^{-1}$$

$$p(0) = \frac{\delta_s}{\lambda_0} + \sum_{l=s+1}^{\infty} \frac{\delta_l}{\lambda_0} \prod_{k=s+1}^{l} \frac{\lambda_{k-1}}{\lambda_k + \delta_k} 1 + \frac{\delta_s}{\lambda_0} + \delta_s + \sum_{l=s+1}^{\infty} \left[1 + \delta_l + \frac{\delta_l}{\lambda_0}\right] \prod_{k=s+1}^{l} \frac{\lambda_{k-1}}{\lambda_k + \delta_k}$$

From these expressions, many of the measures of interest can be derived. For instance, if the total duration of an auction from when it starts to when it ends, excluding the rest time, is denoted by $T = \tau - R$, then

$$p(0) = \frac{1 + \frac{1}{\lambda_0}}{1 + \frac{1}{\lambda_0} + \mathbf{E}\{T\}} \tag{5}$$

so that the expected duration of the auction until a sale occurs is:

$$\mathbf{E}\{T\} = \left[1 + \frac{1}{\lambda_0}\right]\left[\frac{1}{p(0)} - 1\right] \tag{6}$$

Similarly, we are interested in knowing what the average sale price is, and it is given by

$$\mathbf{E}\{S\} = \sum_{l=s}^{\infty} lp(a(l)) = s\delta_s p(s) + p(s) \sum_{l=s+1}^{\infty} l\delta_l \prod_{k=s+1}^{l} \frac{\lambda_{k-1}}{\lambda_k + \delta_k}, \tag{7}$$

so that the average income per unit time for the seller becomes:

$$\phi = \frac{\mathbf{E}\{S\}}{1 + \frac{1}{\lambda_0} + \mathbf{E}\{T\}} \tag{8}$$

2.2 The Effect of Network QoS

We imagine that an "auction centre", i.e. a computer system located somewhere in the Internet which is used by various buyers and sellers to run their economic transactions, is currently used by our seller to run his auction. Bidders then have to access this auction centre via the network when they wish to make a bid. Similarly the seller may be located elsewhere in the Internet and also receives information about bids from the auction centre and then communicates its decision to seal or to wait via the Internet to the auction centre.

As indicated in the introduction, when both bidders and the seller access the auction centre and the software that is running the auction, the network quality of service (QoS), which typically includes packet losses, delay and jitter, will also affect the economic performance of the auction. Thus the network introduces additional effects on the auctions, including:

– Delaying the arrival of the bids due to normal packet travel delays and also potentially because of congestion,
– Reversing the order of some of the bids and thus introducing unfairness with later bidders being taken into account before earlier bidders due to the fact that some bidders' paths through the network may be shorter or less congested than other bidders' paths, which can be exacerbated by "jitter" or significant variance in the packet delays,
– Losing some of the bids due to packet losses so that the price increases are smaller and hence further delayed, and

– Delaying the arrival of information concerning the seller's acceptance of a bid due to normal delays or congestion, including a "round-trip" delay due to the information needing to reach the seller from the auction centre about the most recent bid, and the seller having to convey its decision back to the auction centre.

Such adverse effects can occur when the network is congested and when a given auction centre is very "popular" and hence highly utilised or sought after. Here we will consider the latter two effects. Let us first consider ϕ in the right-hand-side of (4).

Let p_i be the probability that the ith bid is lost by the network before it arrives at the auction centre, and let $\hat{D}_i = D_i + r_i$ be the net decision delay which includes the round-trip delay from auction centre to seller and back. Let the bid inter-arrival times be independent and identically distributed with density without packet losses $f(x)dx = \mathbf{P}[x \leq T_{i+1} - T_i < x + dx]$ with Laplace–Stieltjes Transform (LST) $f^*(s)$. Now if $\{T_i'\}_{i=1}^{\infty}$ is the sequence of effective bid arrival instants excluding those bids which have been lost, if the losses occur in an independent and identically distributed manner with probability and p, then the LST of the density function of the random variable $(T_{i+1}' - T_i')$ representing the effective inter-arrival times of bids is obviously:

$$g^*(s) \equiv \int_0^{\infty} \mathbf{P}[x \leq T_{i+1}' - T_i' < x + dx]e^{-sx} \tag{9}$$

$$= \sum_{j=0}^{\infty} (f^*(s))^{j+1} p^j (1-p) = \frac{(1-p)f^*(s)}{1 - pf^*(s)}, \tag{10}$$

and the expected value is:

$$\mathbf{E}[T_{i+1}' - T_i'] = \frac{\mathbf{E}[T_{i+1} - T_i]}{1-p} \tag{11}$$

In the case of Poisson arrivals, $f(x) = \lambda e^{-\lambda x}$ for $\lambda > 0$, giving $g(x) = (1 - p)\lambda e^{-\lambda(1-p)}$. Just for the sake of evaluating the effect of both packet losses and of delayed decision, suppose that decisions are taken after a constant time D and that the network will delay a decision from reaching the auction centre by a further round-trip constant value r which includes the time it takes the bidder to be informed of the current price of the good, and the time it takes the bid to reach the auction centre. Then a decision *not* to sell is taken with probability $q = \mathbf{P}\{D + r > T_{i+1}' - T_i'\} = 1 - e^{-(D+r)(1-p)\lambda}$ which is the probability that a new bid arrives before the decision to sell is taken, and it includes the effect of loss. If all bids have an average increment of $E[I]$ over the previous bid, and if the expected value of the first bid is $E[I_s]$, then

$$\phi = \frac{\mathbf{E}[I_s] + \frac{q\mathbf{E}[I]}{1-q}}{R + D + r + \frac{q}{1-q}\mathbf{E}\{(T_{i+1}' - T_i')I_{\{r+D>T_{i+1}'-T_i'\}}\}}, \tag{12}$$

where I is the indicator function. Note that before a bid is accepted, a total average number of $1/(1-q)$ bids occur, including the last one. The total auction time includes the time for the acceptance of the final bid which is $(D+r)$, and the time for an average number $q/(1-q)$ bids to arrive and these are all turned down because a new bid arrives before the previous bid can be accepted. The bids that are turned down wait on average for a time $\mathbf{E}\{(T'_{i+1}-T'_i)I_{\{r+D>T'_{i+1}-T'_i\}}\}$ before they are superseded by a higher bid. On the other hand, the average income per auction is obviously:

$$\Upsilon = \mathbf{E}[I_s] + \frac{q}{1-q}\mathbf{E}[I] \tag{13}$$

Clearly, Since

$$\mathbf{E}\{(T'_{i+1}-T'_i)I_{\{r+D>T'_{i+1}-T'_i\}}\} = \frac{q}{\lambda(1-p)} - (1-q)(r+D) \tag{14}$$

we obtain an expression that relates the economic "success" of the auction and the network QoS parameters q and r:

$$\begin{aligned}\phi &= \frac{\mathbf{E}[I_s] + \frac{q}{1-q}\mathbf{E}[I]}{R+D+r+\frac{q^2}{\lambda(1-p)(1-q)} - q(r+D)} \\ &= \frac{(1-q)\mathbf{E}[I_s] + q\mathbf{E}[I]}{R(1-q)+(D+r)(1-q)^2+\frac{q^2}{\lambda(1-p)}}\end{aligned} \tag{15}$$

2.3 The Case When Goods Are Not Always Available for Sale

Up to now, we have implicitly assumed that there is an unlimited backlog of goods to be sold, waiting in the "seller's store room" for a sale to occur. This was equivalent to assuming that during a rest period, a new good is brought to the seller so that the next auction can start after the previous sale is completed and the rest period ends. Here we will modify the assumption, so that goods also arrive to the auction according to an arrival process, and an auction cannot start until there is at least one item for sale available. Thus we will suppose that goods for sale arrive singly to the seller at instants $0 < a_1 < a_2 < \ldots$, so that bids can only start after the good is actually available for sale. We will also define the instants $0 < d_1 < d_2 < \ldots$ when the successive items are sold.

In this case, if we look at things from the point of view of whoever *owns* the goods and is anxious to sell them, a significant measure of interest is $W_j = (d_j - a_j)$ the total time that a good has to wait before it actually ends up being sold. As before, let $\tau_j = \sum_{i=1}^{n(i)} A_{ji} + D_{j,n(j)} + R_j$. We then have:

- $d_1 = a_1 + \tau_1 - R_1$,
- $d_{j+1} = d_j + R_j + \tau_{j+1} - R_{j+1}$ if $a_{j+1} \leq d_j + R_j$,
- $d_{j+1} = a_{j+1} + \tau_{j+1} - R_{j+1}$ if $a_{j+1} > d_j + R_j$.

or equivalently:

- $W_1 = \tau_1 - R_1$,

- $W_{j+1} = W_j + a_j + R_j + \tau_{j+1} - R_{j+1} - a_{j+1}$ if $a_{j+1} \leq d_j + R_j$ or $a_{j+1} - a_j \leq W_j + R_j$,
- $W_{j+1} = \tau_{j+1} - R_{j+1}$ if $a_{j+1} > d_j + R_j$ or $a_{j+1} - a_j > W_j + R_j$.

which gives us an equational form similar to Lindley's equation [7]:

$$W_{j+1} = \tau_{j+1} - R_{j+1} + [W_j + R_j - (a_{j+1} - a_j)]^+ \tag{16}$$

where $[X]^+ = X$ if $X > 0$ and $[X]^+ = 0$ if $X \leq 0$.

3 The Analysis of Sealed Memoryless Bids

Sealed bids differ from auctions in the essential point that the seller does *not know* the amount of each of the bids it receives until it stops the bidding process and it opens the "sealed envelopes" that contain the details of each bid. In some cases, the seller may also not know *how many* bids have been submitted, so that it must make its decision to close a bid and accept the highest bid without knowing either its amount nor the number of bids. This section will be devoted to deriving analytical results for this economic mechanism.

Let $0 < T_1 < T_2 < \ldots$ be the random points of a Poisson process with intensity λ, representing the arrival instants of the bids. Denote by B_i the bid generated at T_i, and assume that the $B_1, B_2, \ldots$ are independent, identically distributed random variables with distribution function

$$F(z) = \mathbf{P}\{B_1 < z\}.$$

Here, the marked Poisson process $\{T_i, B_i\}_{i=1}^{\infty}$ (cf. Karr [5]) models a situation where where there is an infinite buyers' population, and each buyer *bids once in his/her life* and the buyers "don't see" each other, i.e., they have no feedback information on the successful bid, contrary to the model in the previous section, and they therefore generate bids independently of each other.

As previously, the seller has a decision delay time D *which is fixed in advance once and for all*, which is assumed to be an exponentially distributed random variable with parameter $\delta > 0$:

$$G(z) = \mathbf{P}\{D \leq z\} = 1 - e^{-\delta z},$$

and we assume that $\{B_i\}$, $\{T_i\}$ and D are independent. The seller has a minimum sale price s:

$$\mathbf{P}\{B_1 \geq s\} = 1 - F(s) = 1.$$

No sale occurs if there are no bids that arrive before time D. On the other hand, we assume that the seller can observe the bids and accepts a bid that arrives before D; so in this latter case, finding the best strategy is a version of the secretary problem.

As a *lower bound* on the performance of the best strategy, consider the case when the seller accepts the first bid within the decision delay D, and let $\hat{S}$ be the price that is then obtained. We then have

$$\hat{S} = B_1 I_{\{T_1 \le D\}},$$

where I is the indicator function. We can obtain $\mathbf{E}\{\hat{S}\}$ from:

$$\begin{aligned}\mathbf{E}\{\hat{S}\} &= \mathbf{E}\{B_1 I_{\{T_1 \le D\}}\} = \mathbf{E}\{B_1\}\mathbf{E}\{\mathbf{P}\{T_1 \le D \mid T_1\}\} \\ &= \mathbf{E}\{B_1\}\mathbf{E}\{e^{-\delta T_1}\} = \mathbf{E}\{B_1\}\frac{\lambda}{\lambda+\delta}.\end{aligned}$$

An *upper bound* of the best strategy when the seller chooses the successful buyer by selecting the maximum valued bid within the decision delay D can also be obtained as follows. If N_D is the number of bid arrival instants in $[0, D]$:

$$N_D = \#\{n, T_n \le D\}$$

then the price obtained $\tilde{S}$ is:

$$\tilde{S} = \max_{1 \le i \le N_D} B_i.$$

Since $\{B_i\}$, $\{T_i\}$ and D are independent:

$$\mathbf{P}\{N_D = n\} = \int_0^\infty \mathbf{P}\{N_D = n \mid D = t\} dG(t) = \int_0^\infty \frac{(\lambda t)^n}{n!} e^{-\lambda t} dG(t),$$

to obtain $\mathbf{E}\{\tilde{S}\}$, we calculate the tail distribution of $\tilde{S}$ as follows. For any $z \ge s$:

$$\begin{aligned}\mathbf{P}\{\tilde{S} \ge z\} &= \sum_{n=0}^{\infty} \mathbf{P}\{\tilde{S} \ge z \mid N_D = n\}\mathbf{P}\{N_D = n\} \\ &= \sum_{n=0}^{\infty} \mathbf{P}\left\{\max_{1 \le i \le N_D} B_i \ge z \mid N_D = n\right\}\mathbf{P}\{N_D = n\} \\ &= \sum_{n=1}^{\infty} \mathbf{P}\left\{\max_{1 \le i \le n} B_i \ge z\right\}\mathbf{P}\{N_D = n\}\end{aligned}$$

therefore

$$\begin{aligned}\mathbf{P}\{\tilde{S} \ge z\} &= \sum_{n=1}^{\infty}\left(1 - \mathbf{P}\left\{\max_{1 \le i \le n} B_i < z\right\}\right)\mathbf{P}\{N_D = n\} \\ &= \sum_{n=1}^{\infty}(1 - F(z)^n)\int_0^\infty \frac{(\lambda t)^n}{n!} e^{-\lambda t} dG(t) \\ &= \sum_{n=0}^{\infty}(1 - F(z)^n)\int_0^\infty \frac{(\lambda t)^n}{n!} e^{-\lambda t} dG(t) \\ &= 1 - \int_0^\infty e^{-\lambda(1-F(z))t}\delta e^{-\delta t} dt \\ &= 1 - \frac{\delta}{\lambda(1-F(z)) + \delta} \\ &= \frac{\lambda(1-F(z))}{\lambda(1-F(z)) + \delta}.\end{aligned}$$

On the other hand for $0 < z < s$,

$$\begin{aligned}\mathbf{P}\{\tilde{S} \geq z\} &= \mathbf{P}\{\tilde{S} \geq s\} = 1 - \mathbf{P}\{N_D = 0\} \\ &= 1 - \int_0^\infty e^{-\lambda t} dG(t) = 1 - \int_0^\infty e^{-\lambda t} \delta e^{-\delta t} dt = \frac{\lambda}{\lambda + \delta},\end{aligned}$$

implying that

$$\begin{aligned}\mathbf{E}\{\tilde{S}\} &= \int_0^\infty \mathbf{P}\{\tilde{S} \geq z\} dz \\ &= \int_0^s \mathbf{P}\{\tilde{S} \geq z\} dz + \int_s^\infty \mathbf{P}\{\tilde{S} \geq z\} dz \\ &= s \frac{\lambda}{\lambda + \delta} + \int_s^\infty \frac{\lambda(1 - F(z))}{\lambda(1 - F(z)) + \delta} dz.\end{aligned}$$

A simple stopping rule can now be introduced by a sequence of thresholds $v_1 \geq v_2 \geq \ldots$ such that the bid B_i is accepted if it exceeds the threshold v_i and there was no prior bid with this property. Let S be the price attained in this manner. Then:

$$S = \sum_{i=1}^\infty I_{\{B_1 < v_1, \ldots, B_{i-1} < v_{i-1}, B_i \geq v_i\}} I_{\{T_i \leq D\}} B_i.$$

Since the duration of the auction is:

$$\tau = R + \sum_{i=1}^\infty I_{\{B_1 < v_1, \ldots, B_{i-1} < v_{i-1}, B_i \geq v_i\}} \min\{T_i, D\}.$$

Then

$$\mathbf{E}\{S \mid \{T_i\}, D\} = \sum_{i=1}^\infty \prod_{j=1}^{i-1} F(v_j) \int_{v_i}^\infty b dF(b) I_{\{T_i \leq D\}},$$

and

$$\mathbf{E}\{S\} = \sum_{i=1}^\infty \prod_{j=1}^{i-1} F(v_j) \int_{v_i}^\infty b dF(b) \mathbf{P}\{T_i \leq D\},$$

If $Y_i = T_i - T_{i-1}$ denotes the ith inter-arrival time of bids then we have:

$$\begin{aligned}\mathbf{P}\{T_i \leq D\} &= \mathbf{E}\{\mathbf{P}\{T_i \leq D \mid T_i\}\} \\ &= \mathbf{E}\{e^{-\delta T_i}\} \\ &= \mathbf{E}\{e^{-\delta \sum_{j=1}^i Y_j}\} \\ &= (\mathbf{E}\{e^{-\delta Y_1}\})^i \\ &= \left(\frac{\lambda}{\lambda + \delta}\right)^i,\end{aligned}$$

implying that

$$\mathbf{E}\{S\} = \sum_{i=1}^\infty \prod_{j=1}^{i-1} F(v_j) \int_{v_i}^\infty b dF(b) \left(\frac{\lambda}{\lambda + \delta}\right)^i.$$

Similarly,

$$\mathbf{E}\{\tau \mid \{T_i\}, D\} = 1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} F(v_j)(1 - F(v_i)) \min\{T_i, D\},$$

therefore

$$\mathbf{E}\{\tau\} = 1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} F(v_j)(1 - F(v_i))\mathbf{E}\{\min\{T_i, D\}\}.$$

Because:

$$\min\{T_i, D\} = D - (D - T_i)^+,$$

one can verify that

$$\begin{aligned}\mathbf{E}\{\min\{T_i, D\} \mid T_i\} &= \mathbf{E}\{D\} - \mathbf{E}\{(D - T_i)^+ \mid T_i\} \\ &= \mathbf{E}\{D\} - e^{-\delta T_i} \int_{T_i}^{\infty} \delta e^{-\delta(t - T_i)} dt = \frac{1}{\delta}\left(1 - e^{-\delta T_i}\right),\end{aligned}$$

therefore

$$\mathbf{E}\{\min\{T_i, D\}\} = \frac{1}{\delta}\mathbf{E}\left\{1 - e^{-\delta T_i}\right\} = \frac{1}{\delta}\left(1 - \left(\frac{\lambda}{\lambda + \delta}\right)^i\right),$$

and so

$$\mathbf{E}\{\tau\} = 1 + \frac{1}{\delta}\left(1 - \sum_{i=1}^{\infty} \prod_{j=1}^{i-1} F(v_j)(1 - F(v_i))\left(\frac{\lambda}{\lambda + \delta}\right)^i\right).$$

Introducing the notation

$$p_i = \prod_{j=1}^{i-1} F(v_j)(1 - F(v_i))$$

then

$$\mathbf{E}\{S\} = \sum_{i=1}^{\infty} p_i \mathbf{E}\{B \mid B \geq v_i\}\left(\frac{\lambda}{\lambda + \delta}\right)^i$$

and

$$\mathbf{E}\{\tau\} = 1 + \frac{1}{\delta}\left(1 - \sum_{i=1}^{\infty} p_i \left(\frac{\lambda}{\lambda + \delta}\right)^i\right).$$

This leads to the numerical problem of choosing the thresholds $\{v_i\}$, which maximize the income per unit time:

$$\frac{\mathbf{E}\{S\}}{\mathbf{E}\{\tau\}} = \frac{\sum_{i=1}^{\infty} p_i \mathbf{E}\{B \mid B \geq v_i\}\left(\frac{\lambda}{\lambda+\delta}\right)^i}{1 + \frac{1}{\delta}\left(1 - \sum_{i=1}^{\infty} p_i \left(\frac{\lambda}{\lambda+\delta}\right)^i\right)}.$$

4 Conclusions

In this paper, we have considered models of economic activities which can be represented as auctions and as sealed bids. Our purpose has to show how such economic activities can be studied using techniques which are commonly used in computer and network performance analysis and in areas of operations research, such as queueing and inventory theory. We have shown how such models can be constructed from first principles, and how they can lead to analytical solutions which provide insight into price formation and how they can be used for the optimisation of economic performance.

In particular, this paper has extended the previous work of one of the authors to include the impact of network QoS parameters such as message loss and delay, and to show how sealed bids, which differ significantly from auctions in the information that is available both to the parties making the offers and to the decider, can also be analysed with a similar approach. It is hoped that our results can motivate further work by attracting the attention of the computer system performance evaluation community to the study of some of the important "applications" which run on the Internet, namely those which involve automated economic transactions such as networked auctions.

Acknowledgements

We thank the anonymous referees for comments which helped us clarify the contributions of this paper, and also for their detailed comments concerning typos and omissions.

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